BME 6938
Neurodynamics

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Two Dimensional Neuron model-General Representation

$$\frac{dV}{dt} = \frac{I - I_{\mathrm{ch}}(V,t)}{C}$$

$$\frac{dn}{dt} = \frac{n_{\infty} - n(t)}{\tau(V)}$$

$$I_{\mathrm{ch}}(V,t) = g_{\mathrm{ion}} n(t)(V - E_{\mathrm{ion}}) + g_L(V - E_L)$$

Steady State: $$I_{\mathrm{ch}}(V,t) = I_{\infty}(V)$$

Define: $$\mathbf{I}(V,I) = \frac{I - I_{\infty}(V)}{C}$$

Condition for Saddle node bifurcation

- Fixed point $$\mathbf{I}(V, I)|_{V^*, I^*} = 0$$

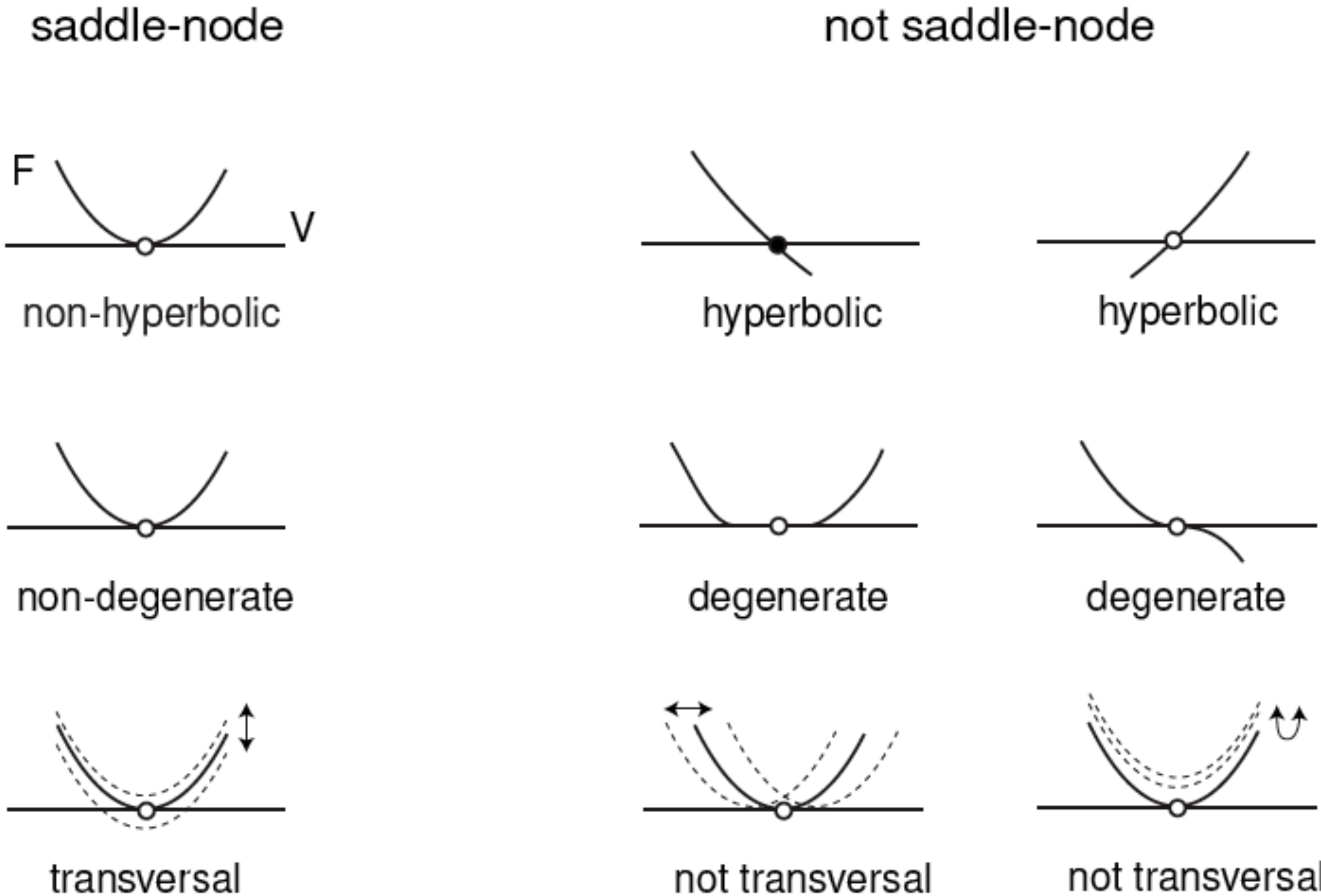
- Non-Hyperbolic $$\left.\frac{\partial \mathbf{I}(V, I)}{\partial V}\right|_{(V^*, I^*)} = 0$$

- Non-Degenerate $$\left.\frac{\partial^2 \mathbf{I}(V, I)}{\partial V^2}\right|_{(V^*, I^*)} \neq 0$$

- Transversal $$\left.\frac{\partial \mathbf{I}(V, I)}{\partial I}\right|_{(V^*, I^*)} \neq 0$$

- **Co-Dim 1 Bifurcation: One Equality Condition that characterizes the bifurcation**

Visual Representation of Saddle Node Bifurcation Constraints



Normal form for Saddle Node Bifurcation

Define

$$a = \frac{1}{2}\ \frac{\partial^2 \mathbf{I}(V,I)}{\partial V^2}\bigg|_{(V^*,I^*)}$$

$$c = \frac{\partial \mathbf{I}(V,I)}{\partial I}\bigg|_{(V^*,I^*)}$$

Normal Form: $$\frac{dV}{dt} = c(I - I^*) + a(V - V^*)^2$$

The $I_{Na,p}$+I_K model

$$
\begin{aligned}
C\frac{dV}{dt} &= I - g_L(V - V_L) - g_{Na}m_\infty(V)(V - V_{Na}) - g_K n(V - V_K) \\
\frac{dn}{dt} &= \frac{n_\infty(V) - n}{\tau(V)}
\end{aligned}
$$

Parameters: C=1; EL=-80; gL=8;gNa=20,gK=10; ENa=60; EK=-90, tau(V)=1

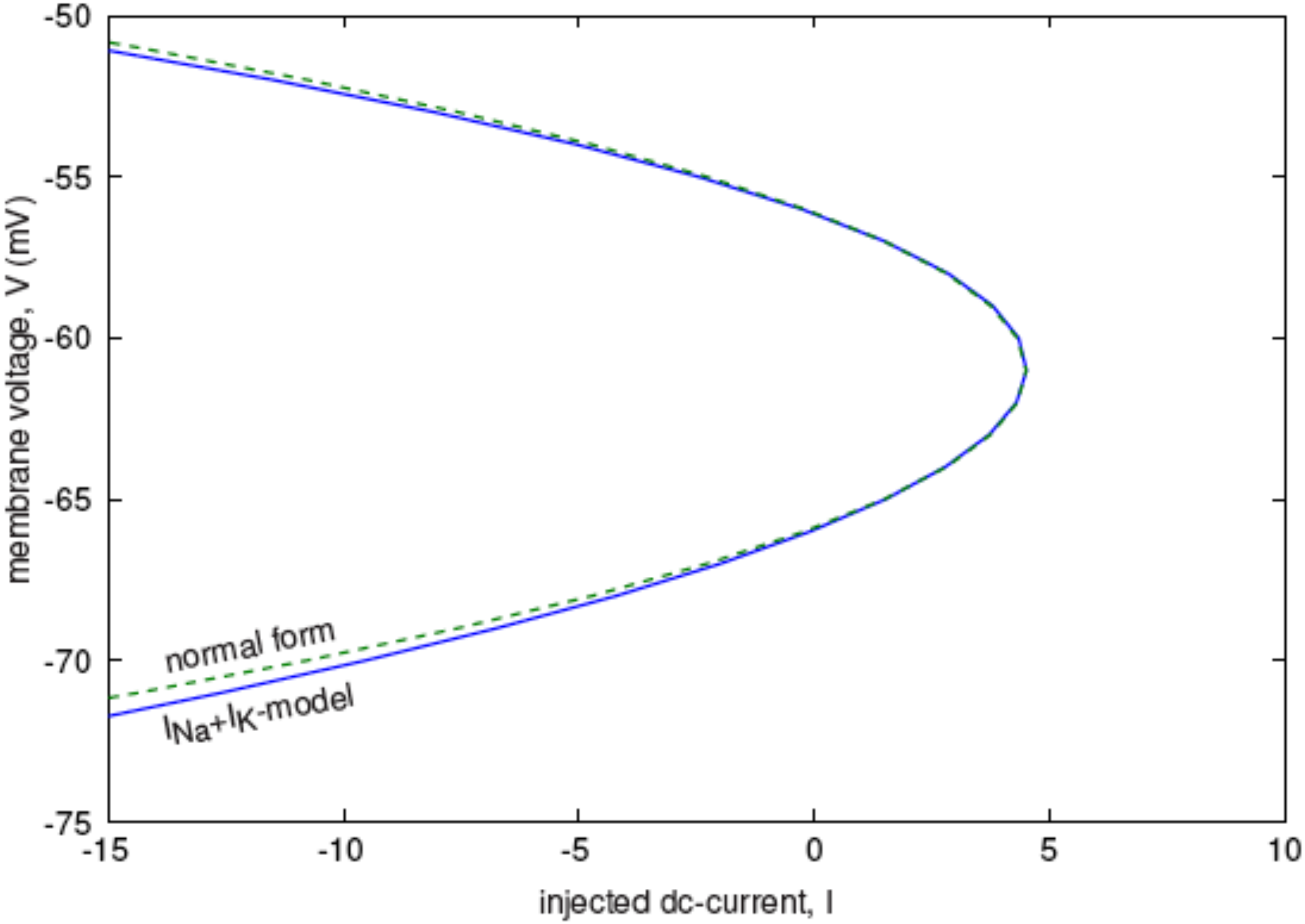
m_∞ has V1/2=-20; K=15

n_∞ has V1/2=-25; K=5

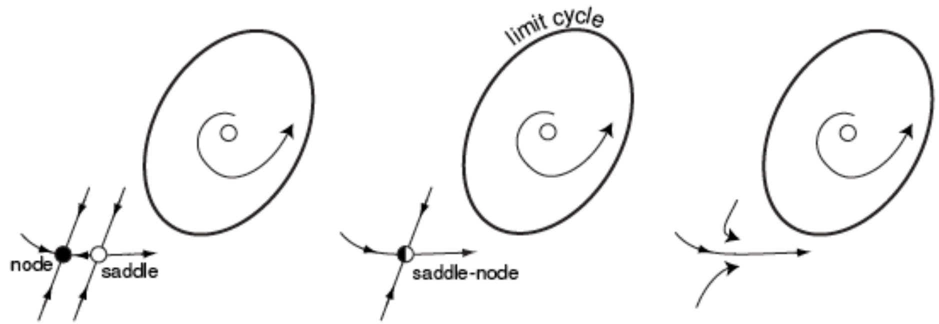
High Threshold K-Currents

Normal Form $$\frac{dV}{dt} = (I - 4.51) + 0.1887(V + 61)^2$$

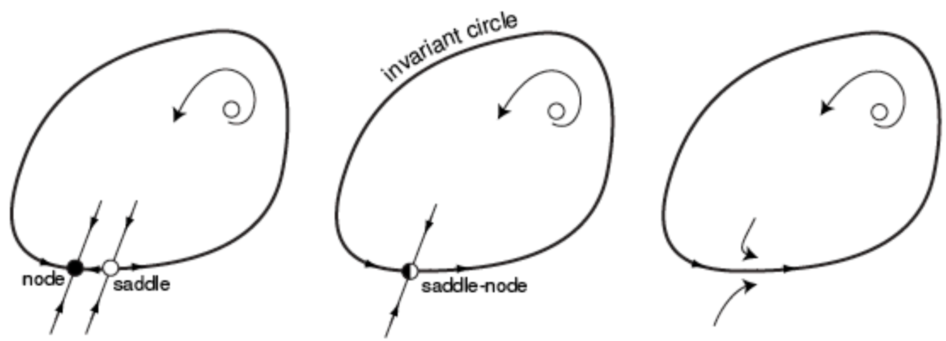
Bifurcation Diagram



Two types of saddle-node bifurcations

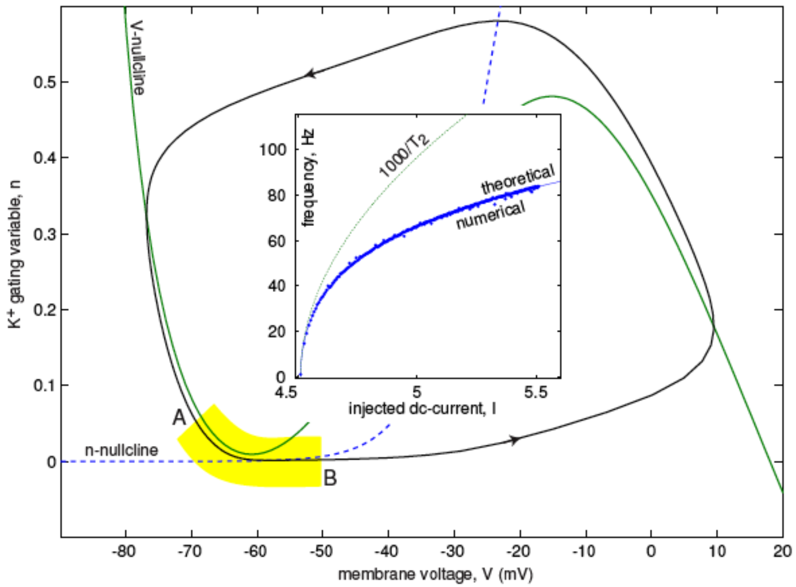


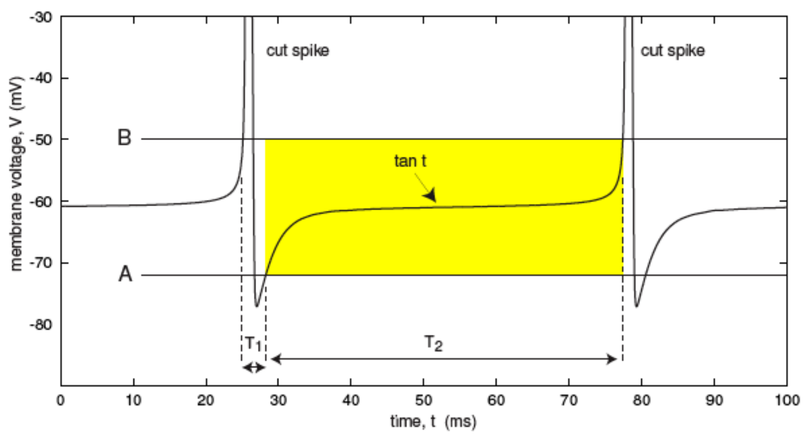
a. saddle-node bifurcation



b. saddle-node on invariant circle (SNIC) bifurcation

Determining the period of spiking





$$T_2 = \frac{\pi}{\sqrt{ac(I - I^*)}} \quad \text{(ms)}$$

$$F = \frac{1000}{T_1 + T_2} \quad \text{(Hz)}$$

Hopf Bifurcation

$$\begin{aligned} \dot{v} &= F(u, v, I) \\ \dot{u} &= G(u, v, I) \end{aligned}$$

$$J = \begin{pmatrix} F_v & F_u \\ G_v & G_u \end{pmatrix}$$

Let the fixed point u=v=0 be the bifurcation point at I=0

The system at the bifurcation point undergoes Hopf Bifurcation if

$$\mathrm{tr}(J) = F_v + G_u = 0$$
$$\omega^2 = \det(J) = F_v G_u - F_u G_v > 0$$

At the fixed point u=v=0 at parameter I=0

- Remember $\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$ $\Delta = \lambda_1 \lambda_2$ $\tau = \lambda_1 + \lambda_2$

Two additional Conditions

Transversality:

Let $c(I) \pm i\omega(I)$ be the complex-conjugate eigen values of J, such that $c(0) = 0$ and $\omega(0) = \omega$. Transversality requires that the real part c(I) must be non-degenerate wrt to I; $\left.\frac{\partial c}{\partial I}\right|_{I=0} \neq 0$

Non-Degeneracy:

Substituting $v = x$ and $F_u u = -F_v x - \omega y$ we have

$$\dot{x} = -\omega y + f(x, y)$$

$$\dot{y} = \omega x + g(x, y)$$

Where $f(x, y) = F(v, u) + \omega y$ and $g(x, y) = -(F_v.F(v, u) + F_u.G(v, u))/\omega - \omega x$

Continued next slide…..

Hopf Bifurcation continued

Define parameter a as follows:

$$a = \frac{1}{16}\left(f_{xxx} + f_{xyy} + g_{xxy} + g_{yyy}\right) + \frac{1}{16\omega}\left[f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy}\right]$$

Non-degeneracy requires $a \neq 0$

If $a < 0$ we have supercritical Hopf bifurcation (appearance of stable limit cycle)

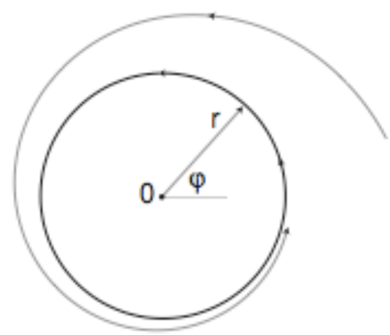
If $a > 0$ we have subcritical Hopf bifurcation (Remember HH model) (disappearance of unstable limit cycle)

- Note: Derivation of the formula for a is given in Guckenheimer and Holmes, 1983

Normal form for Hopf-Bifurcation

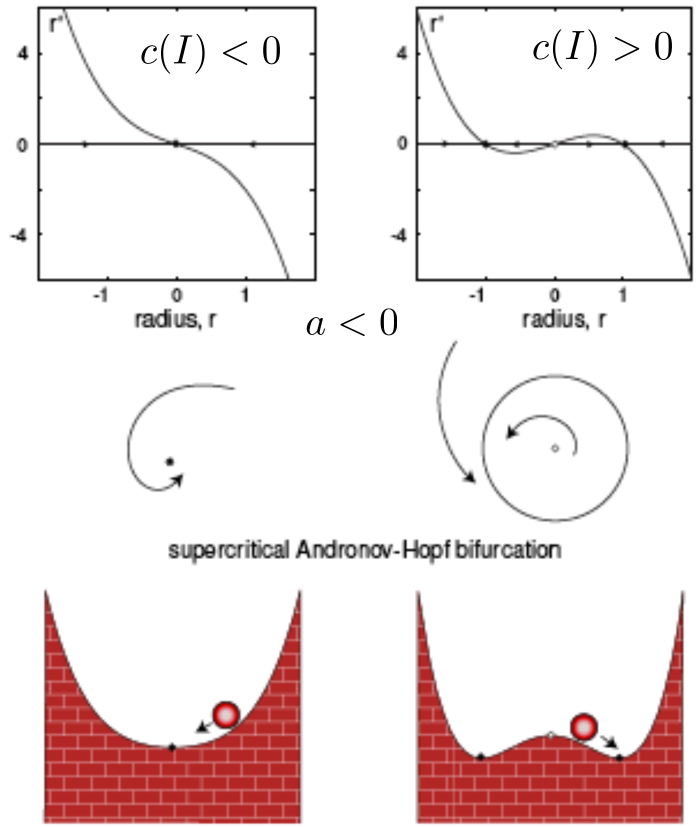
$$\begin{aligned} \dot{r} &= c(I)r + ar^3 \\ \dot{\phi} &= \omega(I) + dr^2 \end{aligned}$$

Where r and ϕ are defined in polar coordinates through figure below



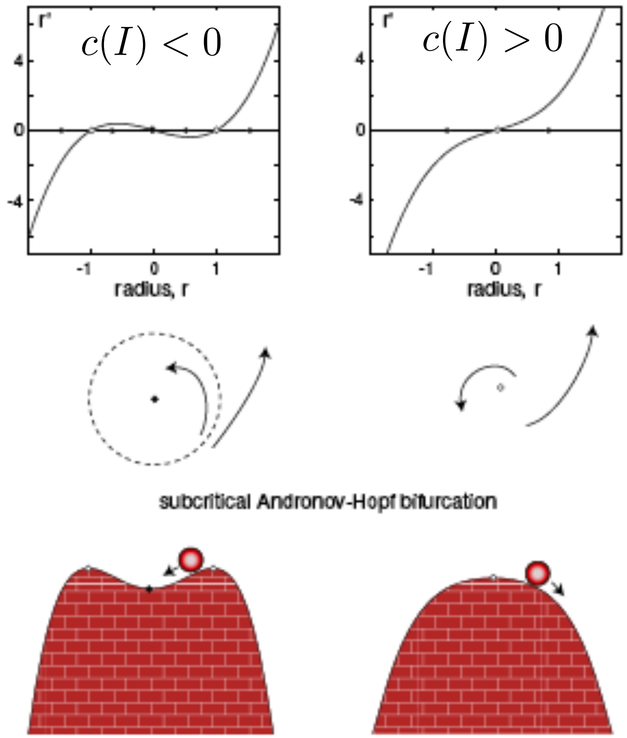
The function ***c(I)*** determines the stability of the equilibrium ***r = 0***. The function ***ω(I)*** determines the frequency of damped or sustained oscillations around this state. The parameter ***d*** describes how the frequency depends on the amplitude. The sign of the non-zero parameter **a** determines the type of Andronov- Hopf bifurcation: see the non-degeneracy condition.

Supercritical Hopf Bifurcation Normal Form



The ODE for **r** variable is similar to the normal form for one-dimensional pitch Fork bifurcation
When c(I)>0, the normal form has a family of stable periodic solutions with amplitude $r = \sqrt{c(I)/|a|}$
and frequency $\omega = \omega(I) + d.c(I)/|a$

Sub-critical Hopf Bifurcation Normal Form

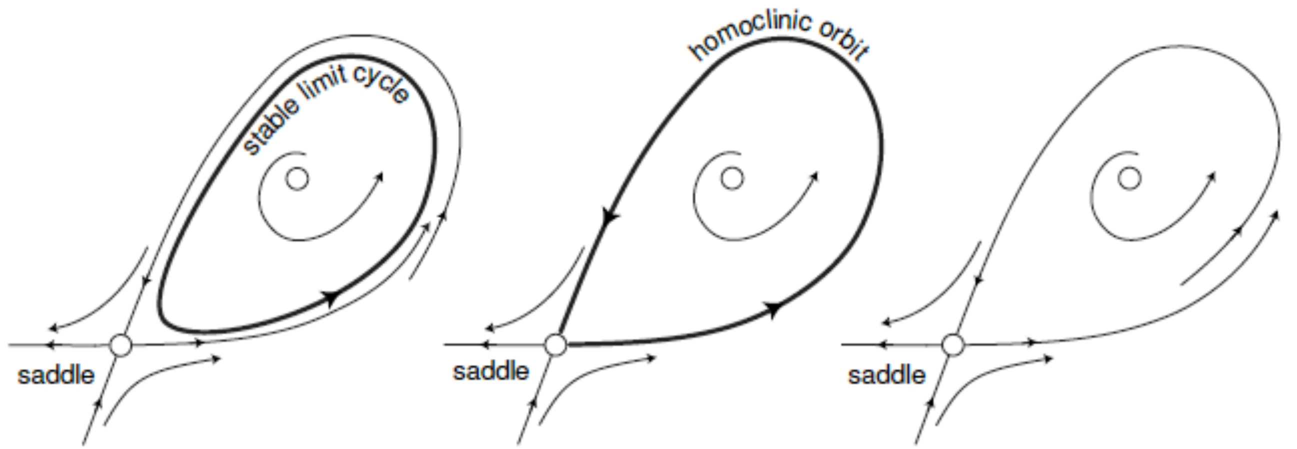


When c(I)<0; there exist pair of unstable fixed points corresponding to unstable Limit cycle with radius $r = \sqrt{c(I)/|a|}$. When c(I)>0 the trajectory diverges from r=0 towards the stable limit cycle present through fold-limit cycle bifurcation

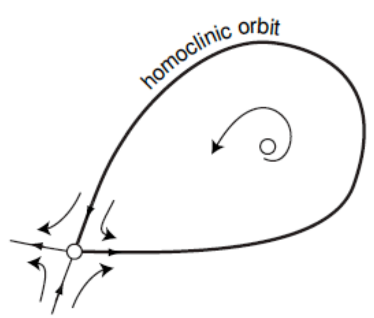
Homoclinic bifurcation-

- Homoclinic bifurcation is similar to saddle node bifurcation in that the I-V curve must be non-monotonic
- The key difference between SNIC bifurcation and homoclinic bifurcation is that the equilibrium is saddle in the later case and it is a saddle node in the former.
- The saddle equilibrium persists after bifurcation while saddle node point vanishes after bifurcation.
- Homoclinic bifurcations are much harder to detect, since they depend on the global properties of the flow and not just the local properties around the bifurcation point.

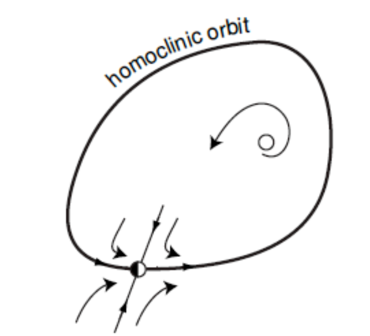
Homoclinic Bifurcation-Visualize



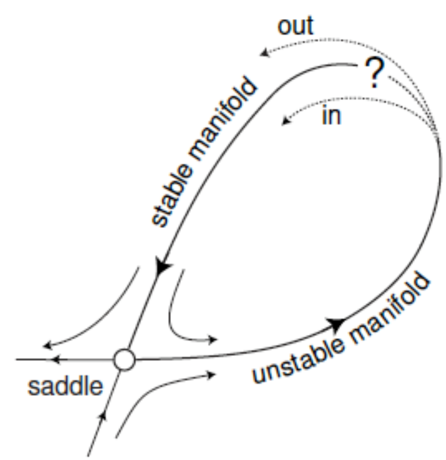
Saddle Quantitiy: $\lambda_1 + \lambda_2 < 0$



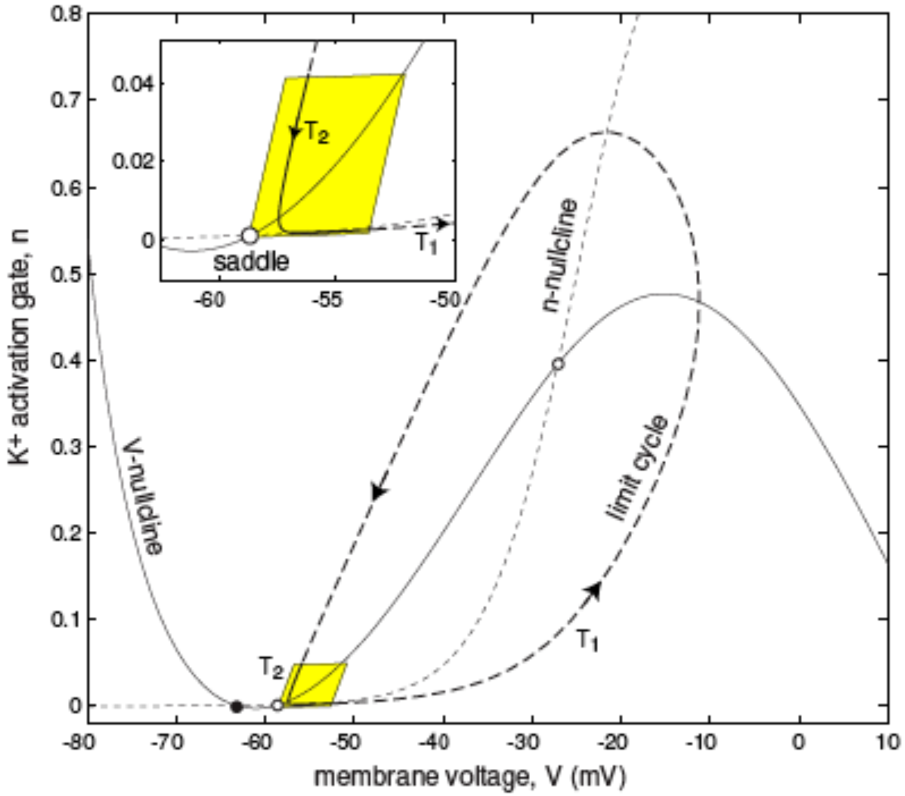
saddle homoclinic orbit bifurcation

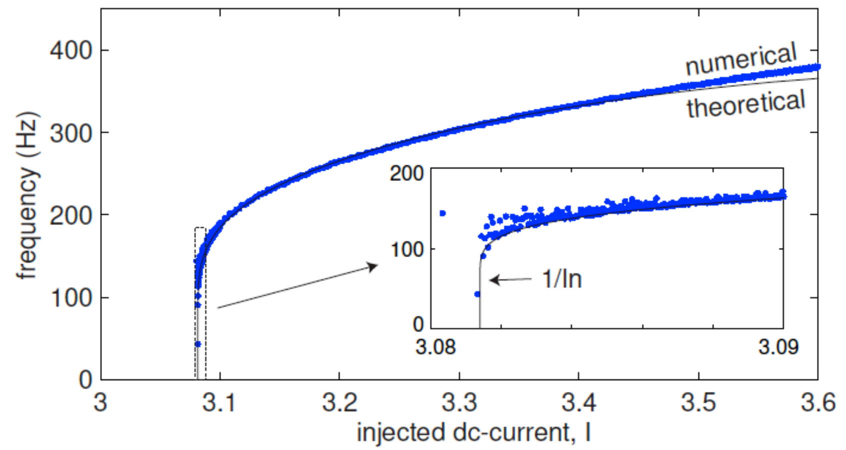


saddle-node on invariant circle bifurcation



Period of Oscillation for spiking generated through homoclinic bifurcation



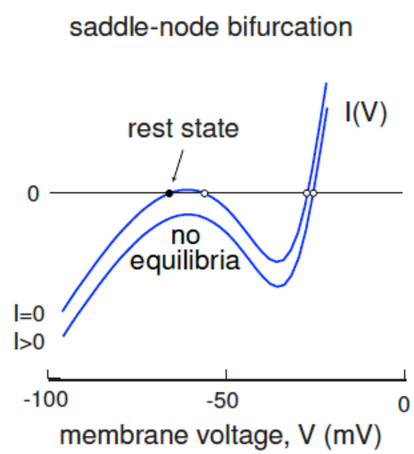


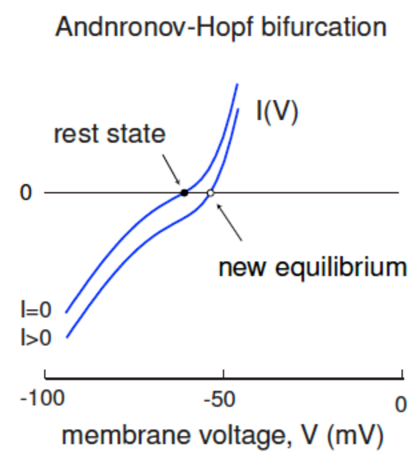
$$T = T_1 + T_2$$

$$T_2 \approx \frac{1}{\lambda_1} \ln \left[\tau (I - I_b) \right]$$

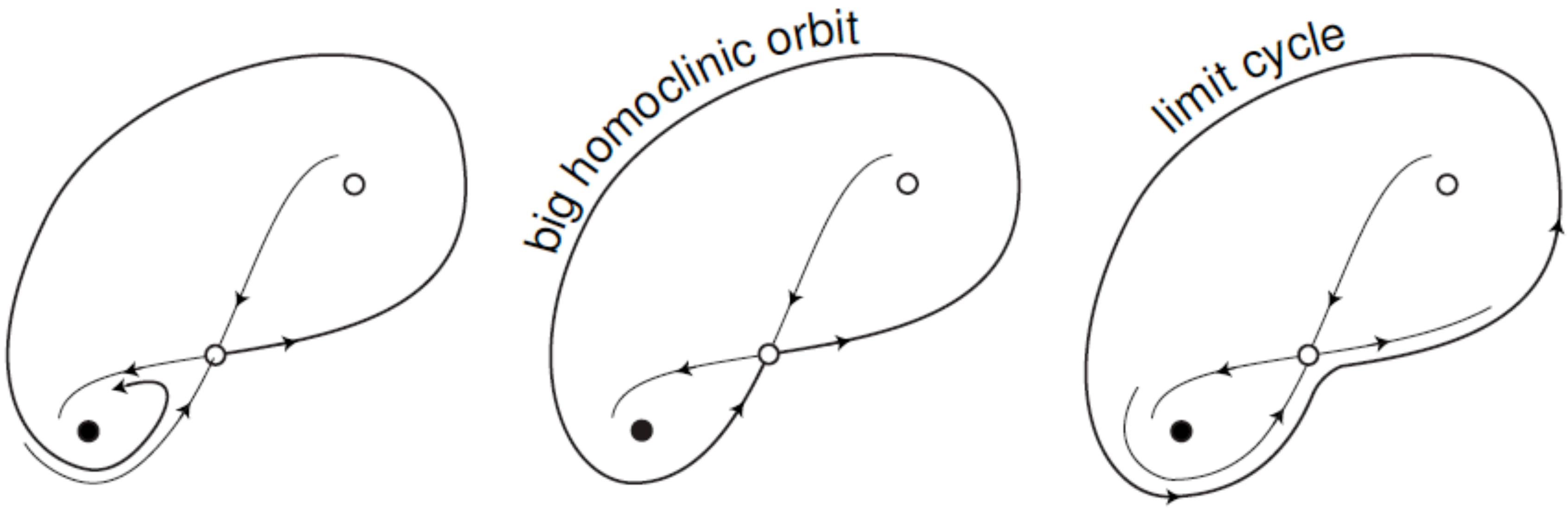
Summary of bifurcation of equilibrium in neuron models

Bifurcation of an equilibrium	fast subthreshold oscillations	amplitude of spikes	frequency of spikes
saddle-node	no	non-zero	non-zero
saddle-node on invariant circle	no	non-zero	$A\sqrt{I - I_b} \to 0$
supercritical Andronov-Hopf	yes	$A\sqrt{I - I_b} \to 0$	non-zero
subcritical Andronov-Hopf	yes	non-zero	non-zero

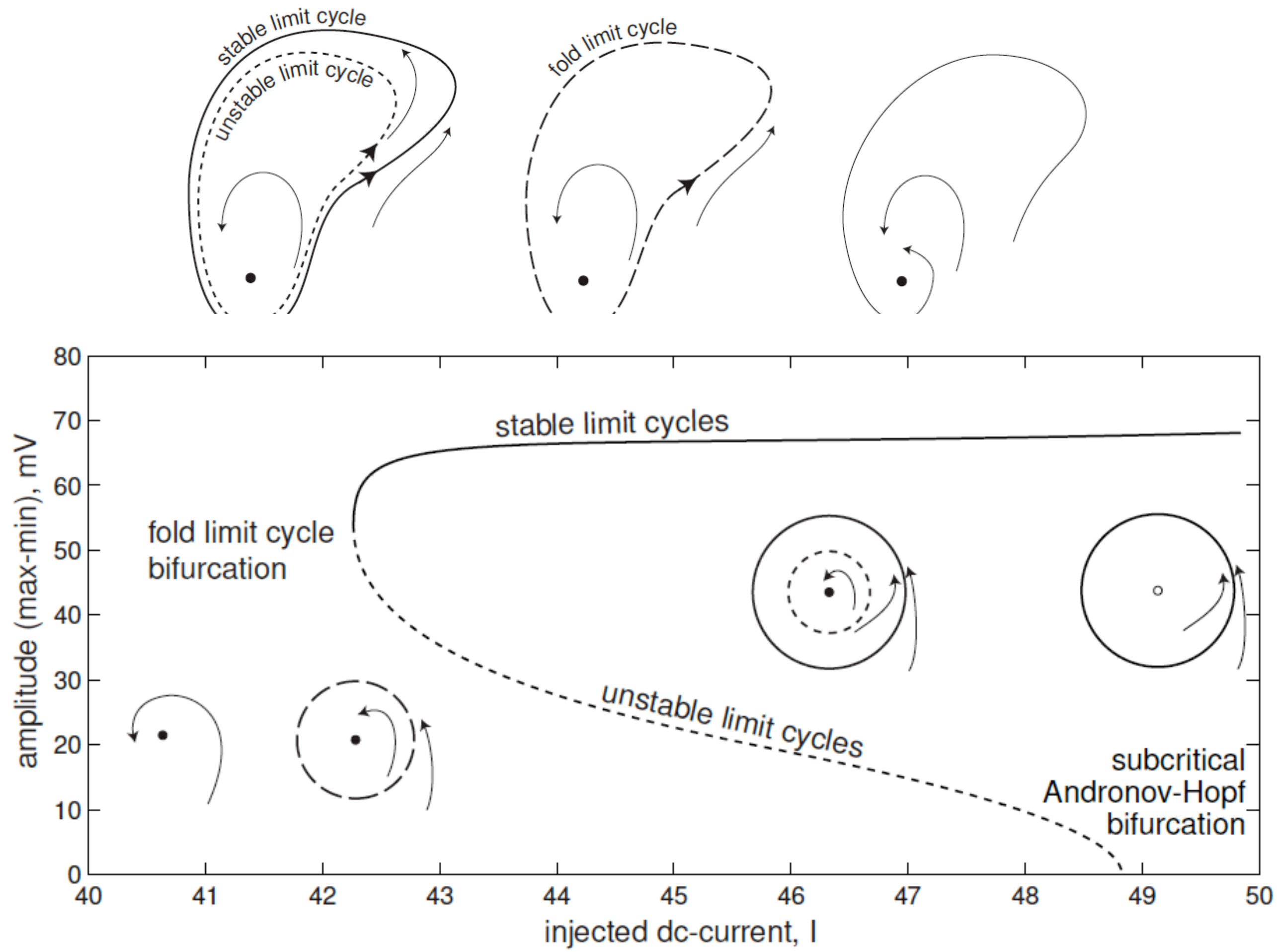




Big-Saddle Homoclinic Bifurcation



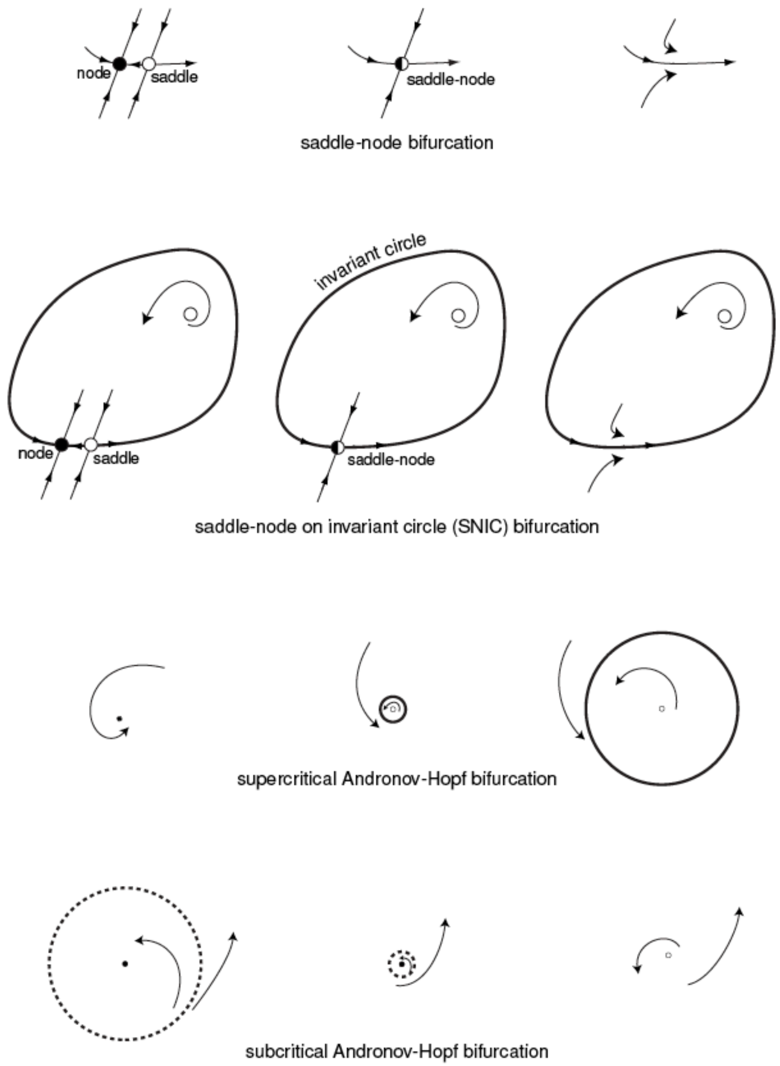
Fold Limit Cycle Bifurcation



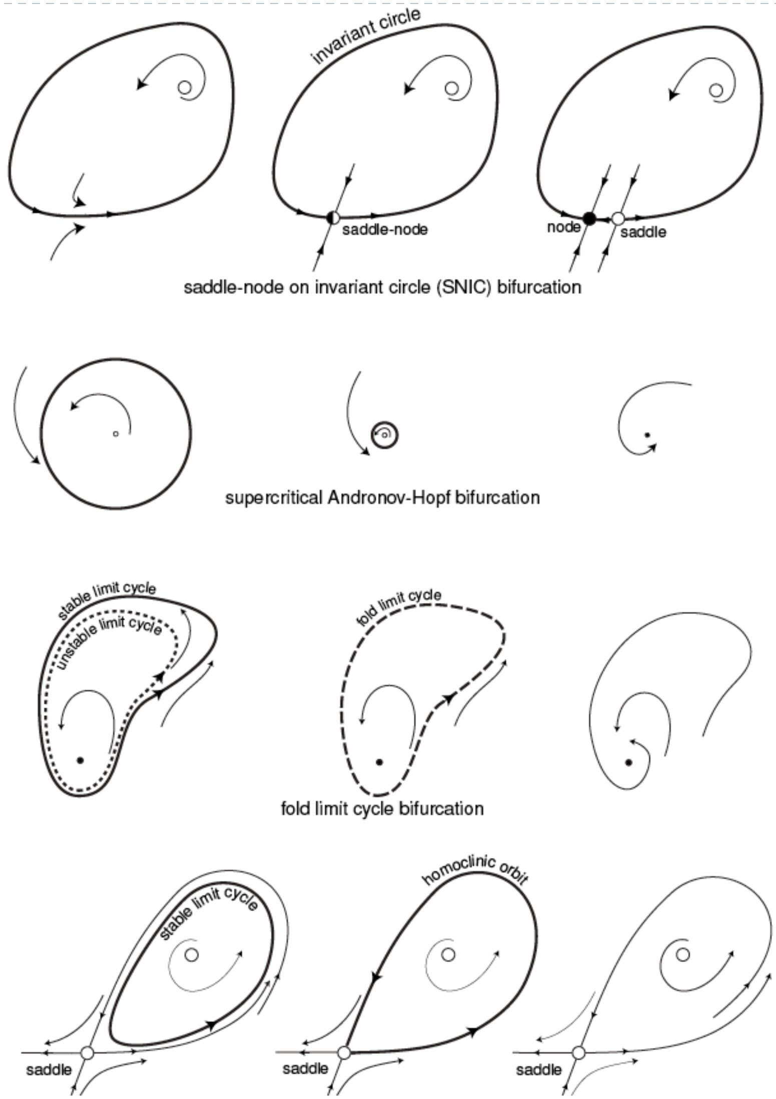
Bifurcation analysis of Fold limit cycle bifurcation can be done using Floquet theory- Not topic for this course

Other option is simple brute force transformation of the dynamics to polar coordinates and demonstrating that the radius under goes saddle node bifurcation- Homework

Summary of all co-dim 1 bifurcations of resting state



Summary of all co-dim 1 bifurcations of spiking state



Other interesting cases

Co-dim 2 bifurcation: Two strict constraints need to be satisfied; Requires change of 2 parameters of the dynamical system (Examples from 2d dynamical systems)

- Cusp Bifurcation
  Intersection of two saddle node bifurcation points.
- Bogdanev-Taken Bifurcation
  Saddle node and Hopf bifuracations occuring simultaneously
- Bautin Bifurcation
  Super and sub critical Hopf bifurcation happening simultaneously
- Saddle node homoclinic Bifurcation
  SNIC occuring together with homoclinic bifurcation

Hard and Soft Loss of Stability

- Bifurcation that results in noticeable change in the dynamics of the system- catastropic and non-reversible cause hard loss of stability

Eg. Sub-critical Hopf Biufurcation; Saddle Homoclinic Bifurcations

- Bifurcation that results in un-noticeable change in the dynamics of the system and the change is reversible

Eg. Supercritical Hopf Bifurcation

Note: SNIC bifurcation is catastrophic but reversible

Example

$$C\frac{dV}{dt} = I - g_L(V - V_L) - g_{Na}m_\infty(V)(V - V_{Na}) - g_K n(V - V_K)$$
$$\frac{dn}{dt} = \frac{n_\infty(V) - n}{\tau(V)}$$

Parameters: C=1; EL=-80; gL=8;gNa=20,gK=10; ENa=60; EK=-90, tau(V)=1

A

m_∞ has V1/2=-20; K=15

n_∞ has V1/2=-25; K=5

Ex3.ode with high threshold K current parameters

Saddle node bifurcation

B

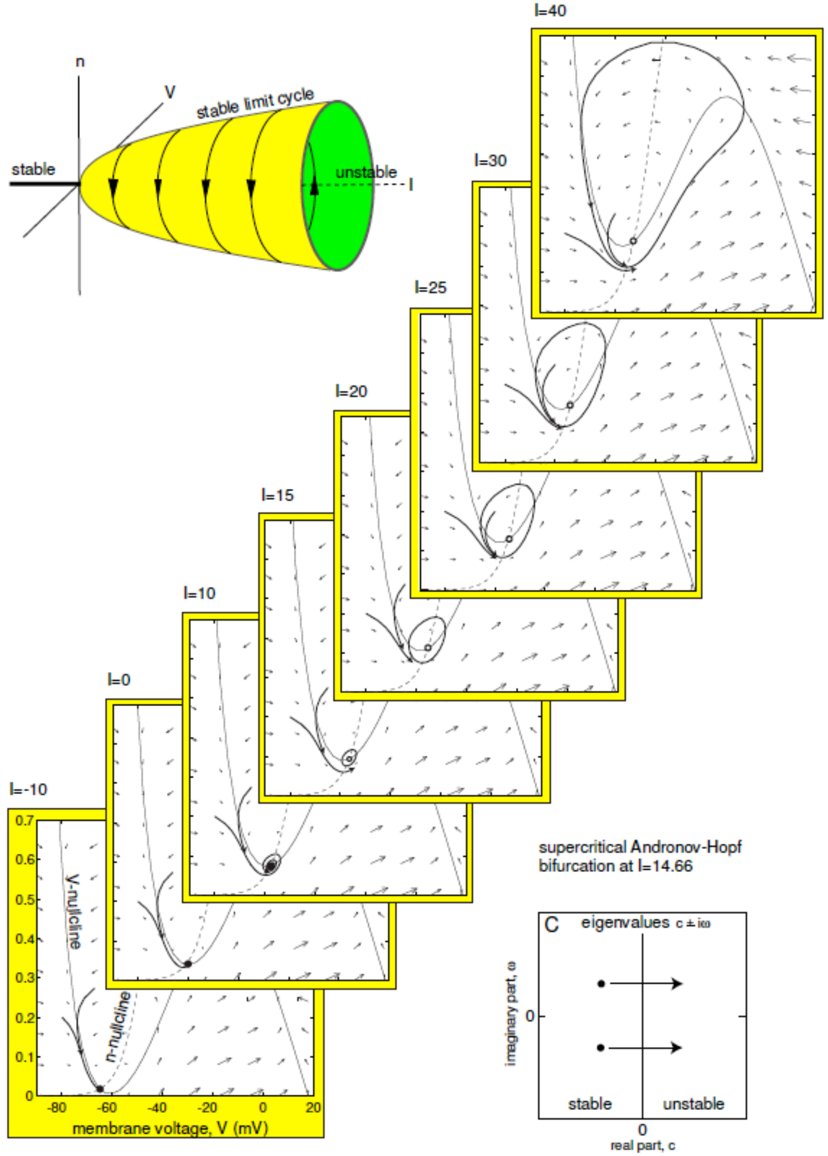
n_∞ has V1/2=-45; K=5

E_L=-78

Ex3.ode with low threshold K current parameters

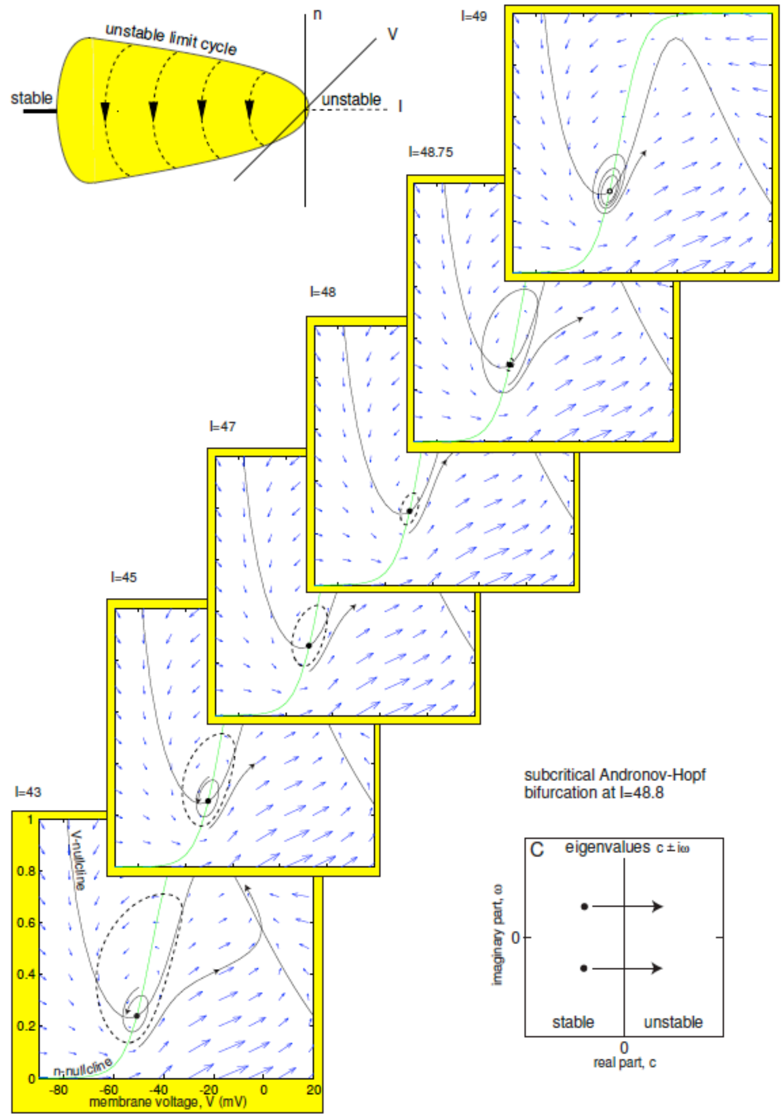
Hopf bifurcation

Hopf Bifurcation



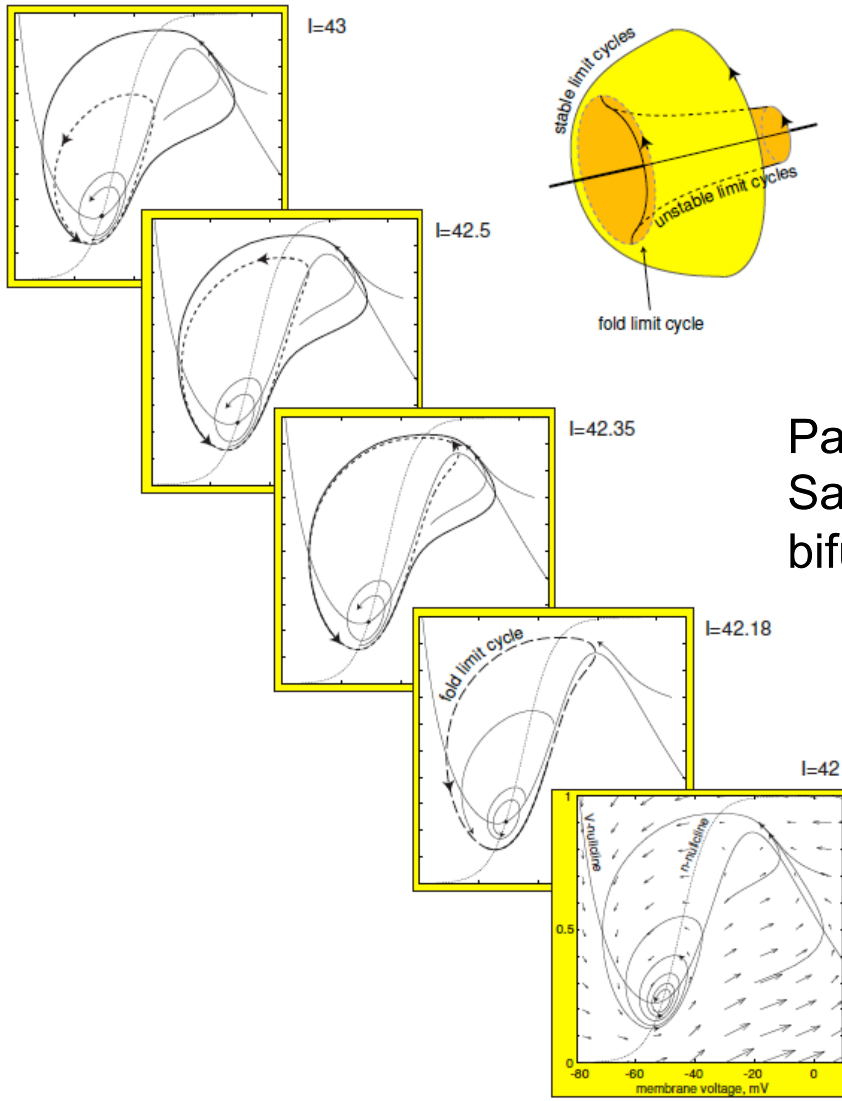
Super-Critical
Parameter- See B

Hopf Bifurcation



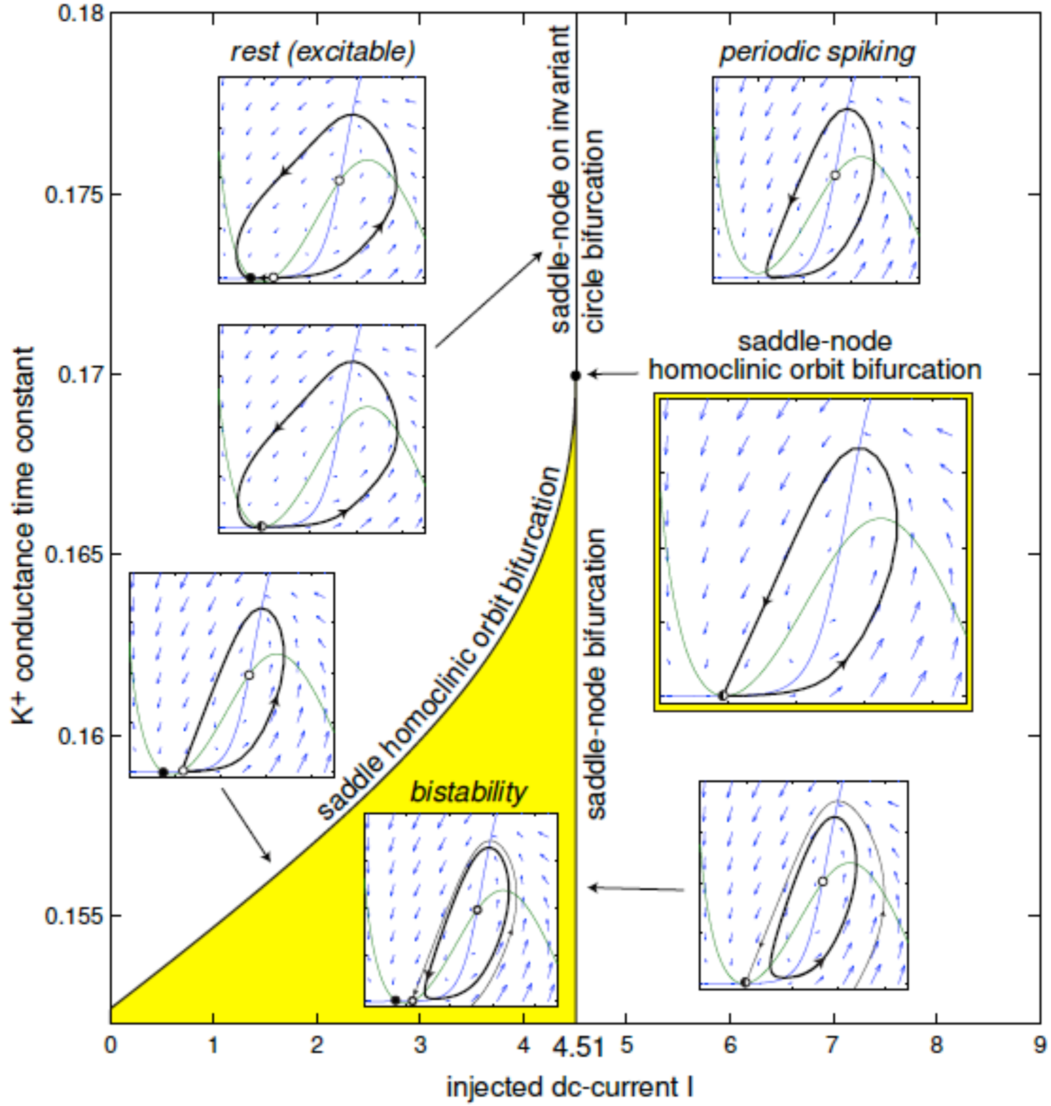
Sub-Critical
Parameter- See B
with the following change
gNa=gK=4; gL=1 and
m_∞ has V1/2=-30; K=7

Fold-Limit Cycle Bifurcation



Parameters-
Same as for sub-critical Hopf bifurcation

Saddle Node Bifurcation-SNIC and Saddle Homoclinic



Parameters- same as A with tau(V)=0.16