Susceptibility for Ising mean-field model

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Here we estimate the susceptibility matrix for a spin-glass model with N neurons, using mean-field approximation. We infer consequences on its spectrum.

1 Self-consistent equations for magnetisation.

In the Ising spin-glass model the potential reads:

$$\phi = \sum_{i=1}^{N} b_i S_i + \sum_{i,j=1}^{N} J_{ij} S_i \, S_j$$

where some J_{ij} can vanish and where $J_{ij} = J_{ji}$. Here $S_i = \pm 1$.

We call $\langle S_i \rangle$ the average of S_i. The susceptibility matrix is:

$$\chi_{ii'} = \frac{\partial \langle S_i \rangle}{\partial b_i'} = \langle S_i S_{i'} \rangle - \langle S_i \rangle \langle S_{i'} \rangle . \quad (1)$$

From the fluctuation-dissipation theorem it is equal to the pairwise correlation.

In what follows $\beta = \frac{1}{k_B T}$ is the inverse temperature and k_B is the Boltzmann constant. As we don't have thermodynamics issues we may take $k_B = 1$ here. $\langle S_i \rangle$ can be approximated with increasing order approximations in terms of self-consistent equations.

- **Mean-Field approximation.**

$$\langle S_i \rangle = \tanh \left[\beta \left(b_i + \sum_j J_{ij} \langle S_j \rangle \right) \right] . \quad (2)$$

- **TAP approximation.**

$$\langle S_i \rangle = \tanh\left[\beta\left(b_i + \sum_j J_{ij}\left[\langle S_j \rangle - J_{ij}(1-\langle S_j \rangle^2)\langle S_i \rangle\right]\right)\right], \quad (3)$$

  where we used $J_{ij} = J_{ji}$.

- **Plefka approximation.** This is an expansion in terms of J_{ij}s and m_is that we should investigate if the previous approximation are not accurate enough.

2 Susceptibility.

We now compute χ in the different approximations.

2.1 Mean-Field approximation

2.1.1 Solution

We have:

$$\chi_{ii'} = \frac{\partial \langle S_i \rangle}{\partial b_{i'}} = \beta\left(1 - \tanh^2\left(b_i + \sum_j J_{ij}\langle S_j \rangle\right)\right)\frac{\partial}{\partial b_{i'}}\left(b_i + \sum_j J_{ij}\langle S_j \rangle\right)$$

$$= \beta\left(1 - \langle S_i \rangle^2\right)\left(\delta_{ii'} + \sum_j J_{ij}\frac{\partial \langle S_j \rangle}{\partial b_{i'}}\right).$$

Introducing the matrices $\mathcal{M} = \beta\, diag\left(1 - \langle S_i \rangle^2\right)$, $\mathcal{I}$ the identity and $\mathcal{J}$ with entries J_{ij}, we obtain:

$$\chi = \mathcal{M}.\left(\mathcal{I} + \mathcal{J}.\chi\right) = \mathcal{M} + \mathcal{M}.\mathcal{J}.\chi$$

giving:

$$\left(\mathcal{I} - \mathcal{M}.\mathcal{J}\right).\chi = \mathcal{M}.$$

This equation has a solution if the matrix $\mathcal{I} - \mathcal{M}.\mathcal{J}$ is invertible. Non invertibility correspond to a change in the number of solutions (controlled actually by β). This condition is given by a condition on the spectrum of $\mathcal{M}.\mathcal{J}$, $Sp(\mathcal{M}.\mathcal{J})$: $1 \in Sp(\mathcal{M}.\mathcal{J})$, i.e. the spectrum contains the eigenvalue 1. This change in the number of solutions correspond to a bifurcation (typically a saddle-node).

Away from the bifurcation points we have:

$$\chi = (\mathcal{I} - \mathcal{M}.\mathcal{J})^{-1}.\mathcal{M}. \tag{4}$$

As $\mathcal{J}$ is symmetric it is diagonalized by a variable change (orthogonal matrix) P where the diagonal for of $\mathcal{J}$, Λ_J is $\Lambda_J = P^* \mathcal{J} P$, where P^* is the transpose, with $P^*.P = \mathcal{I}$. Therefore:

$$\Lambda = P^* \chi P = P^* (\mathcal{I} - \mathcal{M}\mathcal{J})^{-1} P P^* \mathcal{M} P$$

We analyze now several cases.

2.1.2 Specific case. The mean-field ferromagnetic model.

Here the matrix $\mathcal{J} = J\mathcal{I}$ where $J > 0$ is a constant. Then the equation of the susceptibility becomes:

$$\chi = (\mathcal{I} - J\mathcal{M})^{-1}.\mathcal{M}. \tag{5}$$

As $\mathcal{M}$ is diagonal χ is diagonal. This is because, in this model, neurons are independent in the thermodynamic limit. Eigenvalues are therefore:

$$\lambda_i = \frac{\beta(1 - \langle S_i \rangle^2)}{1 - \beta J \left(1 - \langle S_i \rangle^2\right)},$$

where $\langle S_i \rangle$ depends on β. The condition for criticality is $J\left(1 - \langle S_i \rangle^2\right) = 1$ for some i.

Note that $\langle S_i \rangle$ is found by solving the self-consistent equation (2).

2.1.3 The Sherrington-Kirckpatrick model.

Here $\mathcal{J}$ is a random, symmetric matrice with independent entries, Gaussian with mean zero and variance $\frac{J^2}{N}$. From the theory of random matrices (Girko) it is possible to know the distribution of eigenvalues of the matrix $\mathcal{J}\mathcal{M}$. Indeed, $\mathcal{J}\mathcal{M}$ is Gaussian, with mean zero and entries on line i having a variance $\sigma_i^2 = J^2 \mathbb{E}_{\mathcal{J}}\left[(1 - \langle S_i \rangle^2)\right]^2$ where $\mathbb{E}_{\mathcal{J}}[\,]$ denotes the average of $\mathcal{J}$. Thus:

$$\sigma_i^2 = J^2 \left[1 - 2q_i + r_i\right] \tag{6}$$

with:

$$\begin{aligned} q_i &= \mathbb{E}_{\mathcal{J}}\left[\langle S_i \rangle^2\right] = \int_{-\infty}^{+\infty} \tanh^2\left(J\sqrt{q_i}h + b_i\right) \frac{e^{-\frac{h^2}{2}}}{\sqrt{2\pi}}; \\ r_i &== \mathbb{E}_{\mathcal{J}}\left[\langle S_i \rangle^4\right] = \int_{-\infty}^{+\infty} \tanh^4\left(J\sqrt{q_i}h + b_i\right) \frac{e^{-\frac{h^2}{2}}}{\sqrt{2\pi}}. \end{aligned} \tag{7}$$

The rightmost equalities come from the fact that the local field $\eta_i = \sum_{i,j=1}^{N} J_{ij} \langle S_i \rangle$ is, under the law of J_{ij}s, Gaussian with mean zero and variance q_i.

When all external fiels b_i are equal the real eigenvalues of $\mathcal{J}\mathcal{M}$ are distributed, in the thermodynamic limit, according to the Wiener semi-circular law with density:

$$\rho(x) = \frac{2}{\pi\sigma^2}\sqrt{\sigma^2 - x^2}, \quad |x| \leq Jq.$$

Especially, the largest eigenvalue is $\sigma = J\sqrt{[1 - 2q + r]}$. In this case the high temperature condition $\max Sp(\mathcal{M}\mathcal{J}) < 1$ corresponds to the De Almeida-Thouless characterizing the limit of the spin-glass phase.

Above this line the spectral radius of $\mathcal{M}\mathcal{J}$ is < 1 and we obtain χ by expand $(\,\mathcal{I} - \mathcal{M}\mathcal{J}\,)^{-1}$ in series in (5), giving:

$$\chi = \sum_{n=0}^{+\infty} (\mathcal{M}\mathcal{J})^n \mathcal{M} = \mathcal{M} + \mathcal{M}\mathcal{J}\mathcal{M} + \mathcal{M}\mathcal{J}\mathcal{M}\mathcal{J}\mathcal{M} + \dots$$

Note that in general $\mathcal{M}, \mathcal{J}$ do not commute. If all external fiels b_i are equal they do and:

$$\chi = \mathcal{M} + \mathcal{M}^2\mathcal{J} + \mathcal{M}^3\mathcal{J}^2 + \dots$$

Here, if P is the orthogonal variable change diagonalizing J and $\Lambda_{\mathcal{J}}$ the diagonal form of $\mathcal{J}$:

$$P^*\chi P = \mathcal{M} + \mathcal{M}^2\Lambda_{\mathcal{J}} + \mathcal{M}^3\Lambda_{\mathcal{J}}^2 + \dots = (\,I - \mathcal{M}\Lambda_{\mathcal{J}}\,)^{-1}\mathcal{M}$$

which is diagonal. This if μ_i is the i-th eigenvalue of $\mathcal{J}$ and λ_i the i-th eigenvalue of χ:

$$\lambda_i = \frac{\beta\mu_i(1 - \langle S\,\rangle^2)}{1 - \beta\mu_i(1 - \langle S\,\rangle^2)}$$

In this example we see that the spectrum of χ is widely conditioned by the spectrum of $\mathcal{J}$. More generally, this is expressed by eq. (5).

Below the AT line the mean-field approximation (2) is not valid anymore and one has to use (3).