Approximations of integro-difference equations using compact support basis expansions

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We have the evolution equation

$$y_{t+1}(\boldsymbol{s}) = \gamma \int d\boldsymbol{r} K(\boldsymbol{s}, \boldsymbol{r}; \boldsymbol{\theta_s}) y_t(\boldsymbol{r}) + \eta_{t+1}(s)$$

and we assume that we can write or approximate the field and the transition kernel in by using a a set of compact support basis functions thus

$$y_{t+1}(\boldsymbol{s}) = \sum_i x^i_{t+1} \phi_i(\boldsymbol{s})$$
$$K(\boldsymbol{s}, \boldsymbol{r}; \boldsymbol{\theta_s}) = \sum_{i_s \in I_s} \lambda^{i_s}_{\theta_s, I_s} \phi_{i_s}(\boldsymbol{s}) \phi_{i_s}(\boldsymbol{r})$$

where I_s is a set of indices and $\lambda^{i_s}_{\theta_s, I_s}$ are a set of coefficient corresponding to the approximation of $K(\boldsymbol{s}, \boldsymbol{r}; \boldsymbol{\theta_s})$. We will also explore other types of kernel approximations as well say

$$K(\boldsymbol{s}, \boldsymbol{r}; \boldsymbol{\theta_s}) = \sum_{i_s \in I_s} \lambda^{i_s}_{\theta_s, I_s} \phi_{i_s}(\boldsymbol{s} - \boldsymbol{r} + \boldsymbol{\tau}_{\boldsymbol{\theta}_s})$$

and so on.

Given the former assumptions we rewrite the evolution equation as

$$\begin{aligned}
\sum_i x^i_{t+1} \phi_i(\boldsymbol{s}) &= \gamma \sum_i x^i_t \int d\boldsymbol{r} K(\boldsymbol{s}, \boldsymbol{r}; \boldsymbol{\theta_s}) \phi_i(\boldsymbol{r}) + \eta_{t+1}(s) \\
&= \gamma \sum_i x^i_t \sum_{i_s \in I_s} \lambda^{i_s}_{\theta_s, I_s} \phi_{i_s}(\boldsymbol{s}) \int d\boldsymbol{r} \phi_{i_s}(\boldsymbol{r}) \phi_i(\boldsymbol{r}) + \eta_{t+1}(s).
\end{aligned}$$

now we multiply with all ϕ_j and integrate to obtain

$$\sum_i x_{t+1}^i \langle \phi_i(\boldsymbol{s}), \phi_j(\boldsymbol{s}) \rangle = \gamma \sum_i x_t^i \int d\boldsymbol{r} K(\boldsymbol{s}, \boldsymbol{r}; \boldsymbol{\theta_s}) \phi_i(\boldsymbol{r}) + \eta_{t+1}(s)$$
$$= \gamma \sum_i x_t^i \sum_{i_s \in I_s} \lambda_{\theta_s, I_s}^{i_s} \langle \phi_{i_s}(\boldsymbol{s}), \phi_j(\boldsymbol{s}) \rangle \langle \phi_{i_s}(\boldsymbol{r}), \phi_i(\boldsymbol{r}) \rangle + \langle \eta_{t+1}(s), \phi_j(\boldsymbol{s}) \rangle .$$

where we use $\langle \phi_{i_s}(\boldsymbol{r}), \phi_j(\boldsymbol{r}) \rangle = \int d\boldsymbol{r} \phi_{i_s}(\boldsymbol{r}) \phi_i(\boldsymbol{r})$.

By adding the equations for all j we obtain

$$[\langle \phi_i(\boldsymbol{s}), \phi_j(\boldsymbol{s}) \rangle] \boldsymbol{x}_{t+1} = \gamma \left[\sum_{i_s \in I_s} \lambda_{\theta_s, I_s}^{i_s} \langle \phi_i(\boldsymbol{r}), \phi_{i_s}(\boldsymbol{r}) \rangle \langle \phi_{i_s}(\boldsymbol{s}), \phi_j(\boldsymbol{s}) \rangle \right] \boldsymbol{x}_t + [\langle \eta_{t+1}(s), \phi_j(\boldsymbol{s}) \rangle] .$$

Alternatively, we can make use of the property of the nice hat functions that for all node locations $\boldsymbol{s}_i$ we have $\phi_j(\boldsymbol{s_i}) = \delta_{i,j}$. By assuming basis function approximation of the noise term of he form $\eta_{t+1}(s) = \sum_j \epsilon_{t+1}^j \phi_j(\boldsymbol{s})$, where η_{t+1}^j are independent. we can evaluate the RHS for $\boldsymbol{s}_i$, and get

$$x_{t+1}^i = \gamma \sum_j \int d\boldsymbol{r} K(\boldsymbol{s}_i, \boldsymbol{r}; \boldsymbol{\theta}_{\boldsymbol{s}_i}) \phi_j(\boldsymbol{r}) x_t^j + \epsilon_{t+1}^i$$

Now, we make the assumption that the function $K(\boldsymbol{s}_i, \boldsymbol{r}; \boldsymbol{\theta}_{\boldsymbol{s}_i})$ can also be expanded in therm of the basis $\{\phi_i(\boldsymbol{x})\}_i$ to get

$$K(\boldsymbol{s}_i, \boldsymbol{r}; \boldsymbol{\theta}_{\boldsymbol{s}_i}) = \sum_k \lambda_{i,k} \phi_k(\boldsymbol{r}).$$

By using this expansion of the kernel we obtain the finite dimensional dynamical model

$$x_{t+1}^i = \gamma \sum_j \left[\sum_k \lambda_{i,k} \langle \phi_k(\boldsymbol{r}), \phi_j(\boldsymbol{r}) \rangle \right] x_t^j + \epsilon_{t+1}^i.$$

The problem here is that the Markov dependency structure can be destroyed for any "local" kernel expansion for which $K(\boldsymbol{s}_i, \boldsymbol{r}; \boldsymbol{\theta}_{\boldsymbol{s}_i})$ is other than $\sum_k \lambda_{i,i} \phi_i(\boldsymbol{r})$. For example, when we have $K(\boldsymbol{s}_i, \boldsymbol{r}; \boldsymbol{\theta}_{\boldsymbol{s}_i}) = \sum_{k \in \mathcal{N}_i \cup \{i\}} \lambda_{i,k} \phi_k(\boldsymbol{r})$ we have x_{t+1}^i depending all all neighbours of neighbours, which is probably not what we want.