Eigenvalues of 2x2 systems

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Suppose we have a 2x2 matrix:

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

This matrix has characteristic polynomial:

$$Det\left(\mathbf{M} - \lambda\mathbf{I}\right) = 0$$

Notice that this can be rewritten in terms of the trace and the determinant:

$$\begin{aligned} 0 &= Det\left(\mathbf{M} - \lambda\mathbf{I}\right) \\ &= Det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - (a+d)\lambda + ad - bc \\ &= \lambda^2 - Tr(\mathbf{M})\lambda + Det(\mathbf{M}) \end{aligned}$$

Using the quadratic formula:

$$\lambda_1, \lambda_2 = \frac{Tr(\mathbf{M}) \pm \sqrt{Tr(\mathbf{M})^2 - 4Det(\mathbf{M})}}{2}$$

The first thing to observe is that the trace is equal to the sum of the eigenvalues: $Tr(\mathbf{M}) = \lambda_1 + \lambda_2$. The second is that $\lambda_1\lambda_2 = Det(\mathbf{M})$ (check these if you aren't convinced by Mark's lecture).

Let's analyze λ_1 and λ_2 to see how the trace and determinant determine their values. We'll break the space down into four regions: $Tr > 0,\ Det > 0$ (region I), $Tr > 0,\ Det < 0$ (region II), $Tr < 0,\ Det < 0$ (region III), $Tr < 0,\ Det > 0$ (region IV).

I. $Tr > 0,\ Det > 0$. We have two cases, corresponding to whether Tr^2 is less than or greater than $4Det$. This will determine whether or not the term inside the square root is negative. If it is negative, the eigenvalues will have an imaginary part.

   (a) If $Tr^2 < 4Det$, then $\sqrt{Tr^2 - 4Det}$ is imaginary, and the two eigenvalues are given by $\lambda_1, \lambda_2 = \frac{Tr}{2} \pm Qi$ for some real Q. Since the trace is positive, the eigenvalues both have positive real part. The positive real component leads to an unstable equilibrium, and the imaginary components cause solutions to spiral out.

(b) If $Tr^2 > 4Det$, then $\sqrt{Tr^2 - 4Det}$ is real, but smaller than Tr. So both eigenvalues still have positive real part and no imaginary part. As a result, the equilibrium is unstable, but solutions do not spiral.

II. $Tr > 0$, $Det < 0$. If Det is negative, then $\sqrt{Tr^2 - 4Det}$ is real and larger than Tr. As a result, one eigenvalue will be positive and the other negative, and the equilibrium will be a saddle point.

III. $Tr < 0$, $Det < 0$ By the same argument as for region II, we also find a saddle equilibrium.

IV. $Tr < 0$, $Det > 0$. Like in region I, we have two cases. But since the trace is negative, we obtain eigenvalues with negative real part, leading to stable equilibria.

(a) If $Tr^2 < 4Det$, then $\sqrt{Tr^2 - 4Det}$, the eigenvalues have imaginary components, so solutions spiral in.

(b) If $Tr^2 > 4Det$, then $\sqrt{Tr^2 - 4Det}$, the eigenvalues are real and negative, so solutions approach the equilibrium without spiraling.

These results are summarized in the diagram below, from page 104 of Izhikevich's book.

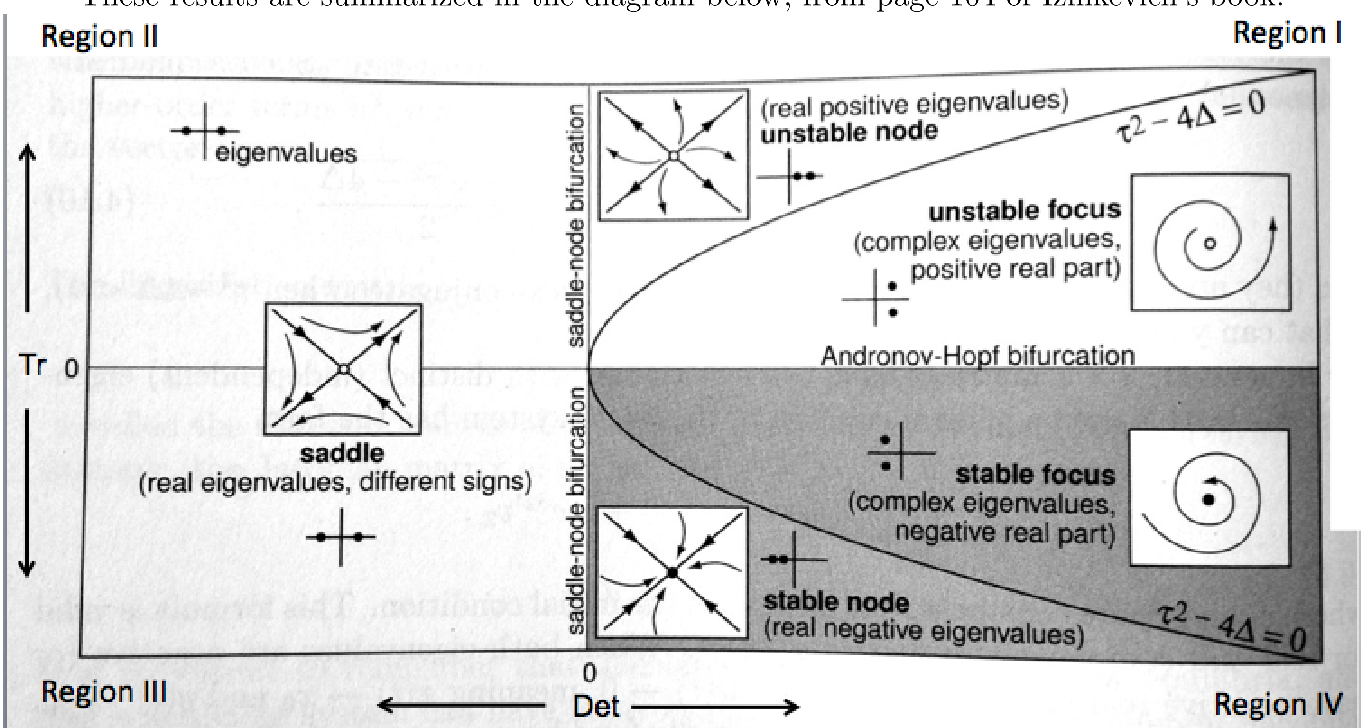

Figure 4.15: Classification of equilibria according to the trace (τ) and the determinan (Δ) of the Jacobian matrix L. The shaded region corresponds to stable equilibria.