Three unbiased estimators of spike-field coherence

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In these notes we show a simplified expression for the Pairwise Phase Consistency (PPC) measure of Vinck et al. (2010). We illustrate it's relation to a bias-corrected spike-field coherence measure from Grasse et al. (2010), and discuss an third notion of spike-field coherence that is intermediate between the two.

In this document, we use the term "event-triggered" rather than "spike-triggered", because we want to apply these measures to neural events other than spikes (e.g. beta-frequency transients in motor cortex).

Pairwise Phase Consistency

Vinck et al. (2010) define Pairwise Phase Consistency (PPC) as the expected dot product between all pairs of (spike-triggered) phase measurements.

$$\hat{\Upsilon} = \frac{2}{N(N-1)} \sum_{j=1..N-1} \sum_{k=j+1..N} \cos(\theta_j - \theta_k)$$

There is an alternative way to express PPC that is faster to calculate, and also reveals a relationship between PPC and similar alternatives. Denote each event-triggered LFP measurement with amplitude r and phase θ as

$$S_j = r_j e^{i\theta_j}.$$

Represent each phase measurement as a phase vector, represented by the complex number z:

$$z_j = S_j/|S_j| = e^{i\theta_j}.$$

Define the average of these phase vectors as $\bar{z}$:

$$\bar{z} = \frac{1}{N} \sum_{j=1..N} z_j.$$

Note: $|\bar{z}|$ is of the same form as Kuramoto's order parameter, where in this case we are averaging phase across event-triggered LFP samples, as opposed to an ensemble of oscillators.

$$|\bar{z}| = \overline{S_j/|S_j|}$$

Consider the expression $|\bar{z}|^2$ and expand it.

$$|\bar{z}|^2 = \Re(\bar{z})^2 + \Im(\bar{z})^2 = \left(\frac{1}{N} \sum_{j=1..N} \Re z_j \right)^2 + \left(\frac{1}{N} \sum_{j=1..N} \Im z_j \right)^2$$

Moving the normalization by N to the left-hand side, and with a bit of algebra...

$$\begin{aligned}
(N|\bar{z}|)^2 &= \left(\sum_{j=1..N} \cos(\theta_j)\right)^2 + \left(\sum_{j=1..N} \sin(\theta_j)\right)^2 \\
&= \sum_{j=1..N} \sum_{k=1..N} \cos(\theta_j)\cos(\theta_k) + \sum_{j=1..N} \sum_{k=1..N} \sin(\theta_j)\sin(\theta_k) \\
&= \sum_{j=1..N} \sum_{k=1..N} \cos(\theta_j)\cos(\theta_k) + \sin(\theta_j)\sin(\theta_k) \\
&= \sum_{j=1..N} \sum_{k=1..N} \cos(\theta_j - \theta_k) \\
&= 2 \sum_{j=1..N-1} \sum_{k=j+1..N} \cos(\theta_j - \theta_k) + N.
\end{aligned}$$

This reveals a relationship between the order parameter $|\bar{z}|$ and the PPC statistic:

$$\frac{N}{2}(N|\bar{z}|^2 - 1) = \sum_{j=1..N-1} \sum_{k=j+1..N} \cos(\theta_j - \theta_k).$$

It follows that PPC can be calculated as

$$\hat{\Upsilon} = \frac{N|\bar{z}|^2 - 1}{N - 1}.$$

Grasse et al. (2010) bias corrected spike-field coherence

The alternative expression for PPC is reminiscent of the bias-corrected event-field coherence from Grasse et al. (modified from Eqn. 10 Grasse et al. 2010)

$$\hat{C} = \frac{N\hat{c} - 1}{N - 1}$$

Where $\hat{c}$ is a biased coherence measurement taken by normalizing the event-triggered phase-amplitude vector by an expected magnitude based on the background LFP power.

$$\hat{c} = \frac{|\bar{S}|^2}{|S_{LFP}|^2}$$

Recalling the definition of the phase and amplitude vector S and how it relates to z:

$$|\bar{z}| = \left|\frac{1}{N} \sum_{j=1..N} z_j \right| = \left|\frac{1}{N} \sum_{j=1..N} \frac{S_j}{|S_j|}\right|,$$

we see that Vinck et al. (2010) and Grasse et al. (2010) are similar. The main difference is that Grasse et al. (2010) normalizes using the average power over all time, whereas Vinck et al. (2010) normalizes each phase vector by the instantaneous amplitude before averaging.

The approach of Grasse et al. (2010) places more weight on LFP samples with higher amplitude. This is useful if LFP phase estimates are contaminated by noise, since higher-amplitude events have better signal-to-noise ratio. However, it still exhibits some bias if fluctuations in LFP amplitude are correlated with changing firing (event) rate. We suggest another statistic that mitigates this below.

A third option

There is a closely related measure where the background power $|S_{LFP}|^2$ is estimated not from the whole-trial LFP, but instead from those times around events. This will remove any spurious coherence arising from coupling between LFP amplitude envelope and event (e.g. spiking) rate.

$$\hat{g} = \frac{|\bar{s}|^2}{\overline{|s|}^2} = \left(|\bar{s}|/\overline{|s|}\right)^2$$

Which, after applying finite-sample-size bias correction, would give

$$\hat{G} = \frac{N\left(|\bar{s}|/\overline{|s|}\right)^2 - 1}{N-1}$$

If we expand the term $|\bar{z}| = \overline{s_j/|s_j|}$ in our expression for the PPC, we see that $\hat{G}$ and $\hat{\Upsilon}$ are very similar.

$$\hat{\Upsilon} = \frac{N\left(\overline{s_j/|s_j|}\right)^2 - 1}{N-1}$$

The main difference is that PPC normalizes phase vectors to unit length before averaging, whereas this approach averages first and then normalizes. Using a local estimate of amplitude, rather than time-average power, may further reduce bias in some applications.

Summary

The PPC measure from Vinck et al. (2010) can be efficiently computed by an expression that resembles Kuramoto's order parameter for coupled oscillators, with the event-triggered LFP phases being our "ensemble" so to speak, followed by a finite sample size bias correction.

This bias correction is identical to that derived in Grasse et al. (2010) for another variation of coherence. The coherence measure outlined in Grasse et al. (2010) can still be biased by a correlation between firing rate and LFP power fluctuations, but normalizing based on the estimated LFP power at the time of each event mitigates this.

Cited

1. Grasse, Dane W., and Karen A. Moxon. "Correcting the bias of spike field coherence estimators due to a finite number of spikes." Journal of neurophysiology 104, no. 1 (2010): 548-558.
2. Vinck, Martin, Marijn van Wingerden, Thilo Womelsdorf, Pascal Fries, and Cyriel MA Pennartz. "The pairwise phase consistency: a bias-free measure of rhythmic neuronal synchronization." Neuroimage 51, no. 1 (2010): 112-122.

1 Three measures of spike-field coherence, cheat sheet:

Unbiased coherence

$$\hat{C} = \frac{N\left(\frac{|\bar{S}|}{|S_{LFP}|}\right)^2 - 1}{N-1}$$

Grasse's unbiased coherence is useful when one is interested in event-field coherence, and either the LFP power is believed to be stationary, or an bias in the coherence estimate caused by correlations between firing rate and LFP power is desired.

Unbiased coherence with local LFP power estimate

$$\hat{G} = \frac{N\left(|\bar{S}|/\overline{|S|}\right)^2 - 1}{N-1}$$

This measure weights examples with higher LFP power more strongly, and normalizes by the LFP power around each event to reduce bias from correlated changes in power and rate.

Pairwise Phase Consistency

$$\hat{\Upsilon} = \frac{N\left(\overline{S_j/|S_j|}\right)^2 - 1}{N-1}$$

Pairwise Phase Consistency is useful when one is interested in a notion of event-field coherence that is not susceptible to correlations between rate and LFP power, and weights all event-triggered LFP samples equally when summarizing the distribution of event-triggered LFP phases.