Calculating the entropy of the Poisson distribution

M. Rule

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The entropy of a Poisson distribution has no closed form. The entropy, in nats, is:

$$\begin{aligned}
H(\lambda) &= -\sum_{k=0..\infty} \Pr(k)\log\Pr(k) \\
&= -\sum_{k=0..\infty} \frac{\lambda^k e^{-\lambda}}{k!}\left[k\log(\lambda) - \lambda - \log(k!)\right] \\
&= \lambda[1-\log(\lambda)] + e^{-\lambda}\sum_{k=0..\infty}\frac{\lambda^k\log(k!)}{k!}
\end{aligned}$$

These notes evaluate some numerical approximations to the entropy.

For high firing rates, the Poisson distribution becomes approximately Gaussian with $\mu = \sigma^2 = \lambda$. Using the forumula for the entropy of the Gaussian, this implies

$$H(\lambda) = \tfrac{1}{2}\log(2\pi e\lambda) + \mathcal{O}\left(\tfrac{1}{\lambda}\right)$$

Compare this to the fist few terms of the series expression, which is increasingly accurate for large λ, but diverges for small λ:

$$H(\lambda) = \frac{1}{2}\log(2\pi e\lambda) - \frac{1}{12\lambda} - \frac{1}{24\lambda^2} - \frac{19}{360\lambda^3} + O\left(\frac{1}{\lambda^4}\right)$$

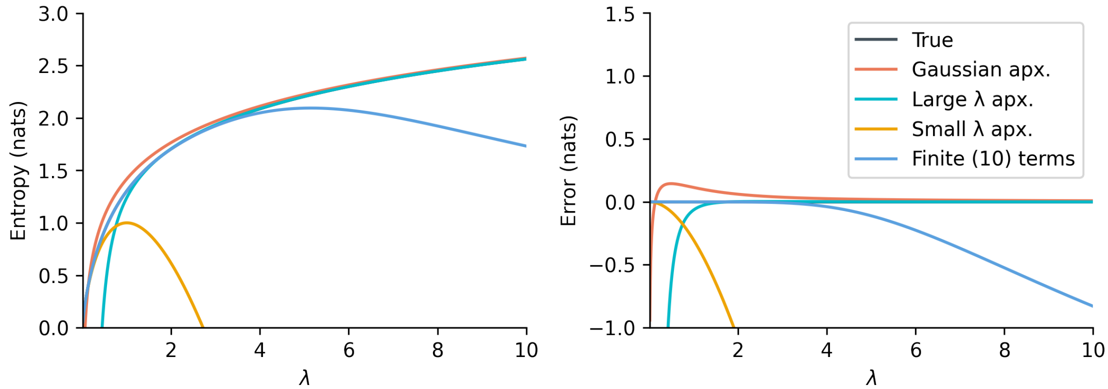
Another way to calculate the entropy for low rates is calculate it using a finite number of terms in the series expansion. For low rates, the probability of $k > \lambda + 4\sqrt{\lambda}$ events is neglegible, so we only need to sum a small number of terms.

$$H(\lambda) \approx -\textstyle\sum_{k=0}^{\lceil\lambda+4\sqrt{\lambda}\rceil} \frac{\lambda^k e^{-\lambda}}{k!}\left[k\log(\lambda) - \lambda - \log(k!)\right]$$

Numerically, you can use the above calculation for $\lambda < 1.78$, taking terms out to $k < \lceil\lambda + 4\sqrt{\lambda}\rceil$. For $\lambda \geq 1.78$, the third-order approximation for large λ becomes accurate.

$$H(\lambda) \approx \begin{cases} -\sum_{k=0}^{\lceil\lambda+4\sqrt{\lambda}\rceil} \frac{\lambda^k e^{-\lambda}}{k!}\left[k\log(\lambda) - \lambda - \log(k!)\right] & \lambda < 1.78 \\ \frac{1}{2}\log(2\pi e\lambda) - \frac{1}{12\lambda} - \frac{1}{24\lambda^2} - \frac{19}{360\lambda^3} & \lambda \geq 1.78 \end{cases}$$

The relative error in this approximation is $|\hat{H} - H|/|H| < 0.0016$



3.0
2.5
2.0
1.5
1.0
0.5
0.0
Entropy (nats)
2
4
6
8
10
λ
1.5
1.0
0.5
0.0
−0.5
−1.0
Error (nats)
True
Gaussian apx.
Large λ apx.
Small λ apx.
Finite (10) terms